Study of Some Elastic Properties for Sandwich Bars with Symmetrical Distribution of Layers

MARIUS MARINEL STANESCU*, DUMITRU BOLCU, ION CIUCA, MIHAELA ELENA BUCULEI

1 University of Craiova, Department of Applied Mathematics, 13 A.I. Cuza, 200396, Craiova, Romania
2 University of Craiova, Department of Mechanics, 165 Calea București, 200620, Craiova, Romania
3 Politehnica University of Bucharest, Department of Materials Science and Engineering, 313 Splaiul Independentei, 060032, Bucharest, Romania

In this paper we determine a field of stress, which verifies the Cauchy equations of equilibrium, the conditions of continuity on the surfaces between layers, and boundary conditions, for a sandwich bar, with symmetrical distribution of layers, subject to traction. We have considered the relationships, previously obtained, for a composite bar built of three layers. Using a mediation formula for deformations and stress, we obtained a new formula for calculating the longitudinal elasticity modulus, in the case in which the constituent materials have different coefficients of transversal contraction. We realized the experimental measurements for test samples from polyesteric resin, reinforced with woven fibers glass, carbon and glass-carbon.

Keywords: composite materials, characteristic curve, elasticity modulus

The composite materials dynamics reserved a special place for sandwich bars made from several overlapped layers with similar thickness. Most studies refer to three layer sandwich bars, the middle layer having visco-elastic behaviour and the inferior and superior layers having extra elastic and resilience proprieties. Other authors having similar studies on the behaviour of these materials suggested the following:
- there is a continuity of displacements and stress between layers;
- there is no deformation along the thickness of the bar;
- the transversal inertial forces are dominant, neglecting longitudinal inertia and rotational inertia of the bar section;
- the external layers have elastic behaviour and are subject to pure bending and the core has elastic or visco-elastic behaviour taking over shear stress;
- the core is not subject to normal stress.

Based on these hypotheses have been developed models considered to be the fundamentals of DTMM theory [5-7]. Considering this theory, it was adapted a variation approach, obtained equations for sandwich plates taking also into consideration different angular deformation for the layers and managing to estimate the stress between the layers [8].

Interlaminar stresses near free edges of composite bar are mainly responsible for delamination failures. Numerous studies have been undertaken to investigate interlaminar stress and failures of laminated composites. In [9] is studied the interlaminar tensile strength under static and fatigue loads including the temperature and moisture effects. In [10] is studied the effect of geometric nonlinearities on free-edge stress fields of bars. In [11] is investigated the response and failure for dropped-ply laminates tested in flat-end compression, and in [12] was shown that the times for delamination onset occurrences in composites can be predicted probabilistically.

Theoretical-obtained results

It is considered a composite bar with a rectangular section of width $2b$, made of $2p + 1$ layers from different materials, with constant thickness along the length of the

* email: mamas1967@gmail.com
bar. Arrangement of layers is considered symmetrical to the median plane. We consider both a mass and elastic symmetry.

We report the bar to a reference system with axes oriented as follows:
- the axe $x$ on the longitudinal direction of bar;
- the axe $y$ on the width direction of the bar;
- the axe $z$ on the thickness direction of bar.

The origin of the reference system is chosen in the median plane of the bar, which is also the median plane of layer, numbered with 1 (fig. 1).

The bar is subject to a load on the longitudinal direction. Because, in general, materials that make up the bar layers have different coefficients of transversal contraction, the state of stress shall be complex. If the length of the bar is high as compared to the transversal dimensions, we can accept that in the middle zone of the bar the stress that occurs does not depend on the coordinated $x$. In these conditions, the equilibrium Cauchy equations have the following form:

\[
\begin{align*}
\frac{\partial \sigma_{xx}}{\partial y} + \frac{\partial \sigma_{yy}}{\partial z} &= 0, \\
\frac{\partial \sigma_{yy}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} &= 0, \\
\frac{\partial \sigma_{zz}}{\partial x} + \frac{\partial \sigma_{xx}}{\partial y} &= 0.
\end{align*}
\]

Due to the symmetry must be satisfied conditions (relative to the axis $y$):

\[
\begin{align*}
\sigma_{yx}(y, z) &= -\sigma_{xy}(y, z), \\
\sigma_{yz}(y, z) &= -\sigma_{zy}(y, z).
\end{align*}
\]

On the lateral surfaces of the bar we have:

\[
\begin{align*}
\sigma_{xy}(b, z) &= 0, \\
\sigma_{yx}(b, z) &= 0.
\end{align*}
\]

Some of these conditions are fulfilled if $\sigma_{yx} = \sigma_{xy} = 0$. The remaining conditions are fulfilled if for the rest of the stress we consider the following representation in mathematical series (for each layer “i”):

\[
\begin{align*}
\sigma_{ii}^{(i)} = \sum_{n=1}^{N} \left[ \frac{2n\pi}{b} \left( 2A_{n}^{(i)}z_{i}^{2} + 2B_{n}^{(i)} \right) \cos \frac{n\pi y}{b} \right], \\
\sigma_{yy}^{(i)} = \sum_{n=1}^{N} \left( 4A_{n}^{(i)}z_{i}^{2} + 2B_{n}^{(i)}z_{i} \right) \sin \frac{n\pi y}{b}, \\
\sigma_{zz}^{(i)} = \sum_{n=1}^{N} \left( -\frac{n\pi}{b} \right) \left( A_{n}^{(i)}z_{i}^{4} + B_{n}^{(i)}z_{i}^{2} + C_{n}^{(i)} \right) \cos \frac{n\pi y}{b}.
\end{align*}
\]

Taking into account the forms of these components of stress tensor, in addition for the normal stress $\sigma_{ss}$ is considered the following form:

\[
\sigma_{ss}^{(i)} = \sum_{n=1}^{N} \int_{0}^{L} f_{n}^{(i)}(z) \cos \frac{n\pi y}{b} + g_{n}^{(i)}(z). \]

On separation surface between the layers “$i$” and “$i+1$” ($i=1, ..., p-1$) must be satisfied the continuity conditions for stress, namely:

\[
\begin{align*}
\sigma_{xx}^{(i)}(z_{i}) &= \sigma_{xx}^{(i+1)}(z_{i}), \\
\sigma_{yy}^{(i)}(z_{i}) &= \sigma_{yy}^{(i+1)}(z_{i}), \\
\sigma_{zz}^{(i)}(z_{i}) &= \sigma_{zz}^{(i+1)}(z_{i}).
\end{align*}
\]

On the exterior surface of plate, the stress must be null, namely:

\[
\begin{align*}
\sigma_{xx}^{(p)}(z_{p}) &= 0, \\
\sigma_{xy}^{(p)}(z_{p}) &= 0, \\
\sigma_{yx}^{(p)}(z_{p}) &= 0.
\end{align*}
\]

The continuity conditions for stress on the separation surface between the layers lead to the following relations:

\[
\begin{align*}
2A_{n}^{(i)}z_{i}^{2} + B_{n}^{(i)} = 2A_{n}^{(i+1)}z_{i}^{2} + B_{n}^{(i+1)}, \\
A_{n}^{(i)}z_{i}^{4} + B_{n}^{(i)}z_{i}^{2} + C_{n}^{(i)} = A_{n}^{(i+1)}z_{i}^{4} + B_{n}^{(i+1)}z_{i}^{2} + C_{n}^{(i+1)},
\end{align*}
\]

and on the exterior surface:

\[
\begin{align*}
2A_{n}^{(p)}z_{p}^{2} + B_{n}^{(p)} &= 0, \\
A_{n}^{(p)}z_{p}^{4} + B_{n}^{(p)}z_{p}^{2} + C_{n}^{(p)} &= 0.
\end{align*}
\]

it results that:

\[
\begin{align*}
B_{n}^{(p)} &= -2A_{n}^{(p)}z_{p}^{2}, \\
C_{n}^{(p)} &= A_{n}^{(p)}z_{p}^{4}, \\
B_{n}^{(i)} &= -2A_{n}^{(i)}z_{i}^{2} - 2 \sum_{k=1}^{i-1} A_{n}^{(k)}(z_{k}^{2} - z_{k+1}^{2}), \\
C_{n}^{(i)} &= A_{n}^{(i)}z_{i}^{4} + \sum_{k=1}^{i-1} A_{n}^{(k)}(z_{k}^{2} - z_{k+1}^{2}), i = 1, p - 1.
\end{align*}
\]

The state of stress is determined with Hooke’s law (it is considered that each of the layers behaves linearly elastic), more precisely:
On the separation surface between the layers \(i\) and \(i+1\) \((i = 1, \ldots, p-1)\), must be satisfied the continuity conditions for deformations, namely:

\[
\begin{align*}
ev_x^{(i)} (z_i) & = e_x^{(i)} (z_i), \\
e_y^{(i)} (z_i) & = e_y^{(i)} (z_i), \\
e_y^{(i)} (z_i) & = e_y^{(i)} (z_i). \\
\end{align*}
\]

From the continuity conditions for deformations on the separation surfaces between the layers it results that:

\[
g_x^{(i)} (z_i) = \frac{\Delta_i}{\Delta} \left[ (\text{C}_{11}^{(i+1)} - \text{C}_{11}^{(i)}) (12A_{i+1} x_{i+1}^2 + 2B_{i+1} x_{i+1}) - \\
\left( \text{C}_{11}^{(i+1)} - \text{C}_{11}^{(i)} \right) (12A_i x_i^2 + 2B_i x_i) \right]. \\
\]

\[
g_y^{(i)} (z_i) = \frac{\Delta_i}{\Delta} \left[ (\text{C}_{12}^{(i+1)} - \text{C}_{12}^{(i)}) (12A_{i+1} x_{i+1}^2 + 2B_{i+1} x_{i+1}) - \\
\left( \text{C}_{12}^{(i+1)} - \text{C}_{12}^{(i)} \right) (12A_i x_i^2 + 2B_i x_i) \right]. \\
\]

\[
f_x^{(i)} (z_i) = \frac{\Delta_i}{\Delta} \left[ (\text{C}_{11}^{(i+1)} - \text{C}_{11}^{(i)}) (12A_{i+1} x_{i+1}^2 + 2B_{i+1} x_{i+1}) - \\
\left( \text{C}_{11}^{(i+1)} - \text{C}_{11}^{(i)} \right) (12A_i x_i^2 + 2B_i x_i) \right] - \\
\frac{n\pi}{\Delta} \left[ (\text{C}_{11}^{(i+1)} - \text{C}_{11}^{(i)}) (A_i x_i^4 + B_i x_i^2) + C_{ii}^{(i)} \right] - \\
\frac{n\pi}{\Delta} \left[ (\text{C}_{11}^{(i+1)} - \text{C}_{11}^{(i)}) (A_{i+1} x_{i+1}^4 + B_{i+1} x_{i+1}^2) + C_{ii}^{(i+1)} \right] - \\
\frac{n\pi}{\Delta} \left[ (\text{C}_{11}^{(i+1)} - \text{C}_{11}^{(i)}) (A_{i+1} x_{i+1}^4 + B_{i+1} x_{i+1}^2) + C_{ii}^{(i+1)} \right]. \\
\]

The bar built from three layers

For a bar built from three layers \((p=2)\) are obtained:

\[
\text{E}_i = \frac{\sum_{i=1}^{p} \sigma_0^{(i)} (z_i) dz}{\sum_{i=1}^{p} e_0^{(i)} (z_i) dz}. \\
\]

The bar built from three layers

If the layers which form the bar are homogenous and isotropic, then:

\[
\text{E}_i = \text{c}_{11}^{(i)} / \text{c}_{11}^{(i)} = 1 / E_i, \\
\]

where \(E\) is Young modulus of the material from layer \(i\), and \(v\) is the Poisson coefficient of the material from layer \(i\). If we neglect the higher order terms, the elasticity modulus (on longitudinal direction), calculated with relation (18) is:

\[
E_i = \frac{1}{E_i} h_i (z_i) z_i + h_i (z_i) (z_i - z_i), \\
\]

in which \(C_{11}^{(i)}, C_{12}^{(i)}, C_{13}^{(i)}, C_{22}^{(i)}\) and \(C_{11}^{(i+1)}, C_{12}^{(i+1)}, C_{13}^{(i+1)}, C_{22}^{(i+1)}\) are elastic constants for the material \(i\) and \(i+1\) respectively. For the layers 2,3,\ldots,p-1, the functions \(g_x^{(i)} (z_i)\) and \(f_x^{(i)} (z_i)\) must be uneven in \(z_i\) variable. Without losing the generality, the functions \(g_x^{(i)} (z_i)\) and \(f_x^{(i)} (z_i)\) can be considered constants. In this case, only two of them are independent. In particular these can be \(A_0^{(i)}\) and \(A_2^{(i)}\).
We note
\[ x = \frac{A_{1}}{A_{2}}. \] (22)

Parameter characterizes the way in which stress is distributed in section.

Using the notations:
\[ \frac{E_{1}}{E_{2}} = e, \] (23)

\[ \frac{z_{1}}{z_{2}} = V \] (is volumetric proportion of the layer 1)
we obtain for elasticity modulus \( E_{L} \), the following expression
\[ \frac{E_{L}}{E_{1}} = \frac{e \cdot \eta \cdot p_{1}(\eta, V, x) + (1 - \eta) \cdot p_{2}(\eta, V, x)}{e \cdot \eta \cdot p_{1}(\eta, V, x) + (1 - \eta) \cdot p_{2}(\eta, V, x)} \] (24)
where
\[ p_{1}(\eta, V, x) = e \cdot (3\eta - 1)(1 - \eta) - (1 - \eta) \cdot (2 \cdot \eta \cdot V - 1 - \eta) \]
\[ p_{2}(\eta, V, x) = e \cdot (1 - \eta) \cdot (3\eta - 1) - (1 - \eta) \cdot (2 \cdot \eta \cdot V - 1 - \eta) \] (25)

We can check immediately that if \( V = 1 \) (namely the bar is built just from material 1), it results that \( E_{L}/E_{1} = 1 \), and hence \( E_{L} = E_{1} \). If \( V = 0 \) (namely the bar is built just from material 2) it results that \( E_{L}/E_{1} = 1/e \), and hence \( E_{L} = E_{2} \).

In the case in which the coefficients of transversal contraction are equal, we obtain:
\[ \frac{E_{L}}{E_{1}} = V + \frac{1-V}{e}, \] (26)

which is the classical formula from calculating the longitudinal elasticity modulus for unidirectional composite materials.

Therefore, the relationship for calculating the elasticity modulus given by (24) is consistent with both the basic physical considerations (if the bar is built just from material 1, then elasticity modulus given by (24), coincides with the elasticity modulus of the material from which is built the bar), as well as with the classical relationship for calculating the longitudinal elasticity modulus for composite materials.

In figure 2 is presented the variation of the \( E_{L}/E_{1} \) ratio for \( \nu_{2} = 0.2; \nu_{1} = 0.3; x = 0.05; e = 0.05 \) respectively \( x = 0.1 \).

In figure 3 is presented the variation of the ratio \( E_{L}/E_{1} \) for \( \nu_{2} = 0.2; \nu_{1} = 0.3; x = 0.05; e = 0.05 \) respectively \( x = 1 \).
It is noted that the factor $x$ has an insignificant influence, which can be verified even for $x=1$.

**Experimental determinations**

There were made four sets of samples as follows:
- the set of sample 1 built from polyester resin (with strengthener 3%);
- the set of sample 2 built from polyester resin reinforced with fiberglass fabric (two layers) with $V=0.82$;
- the set of sample 3 built from polyester resin reinforced with carbon fiber fabric (two layers) with $V=0.88$;
- the set of sample 4 built from polyester resin reinforced in the exterior layers with carbon fiber fabric (two layers) and in the median layer reinforced with fiberglass fabric (two layers) with $V=0.5$.

Volumetric proportions were determined by weighing the composite plates, and in parallel, by weighing the fibers which are into the composition of those plates, we obtained first of all the mass proportions of the components.

In figure 4 is presented the characteristic curve for a sample from the first set of samples, in figure 5 is given the characteristic curve for a sample from the second set of samples, in figure 6 is given the characteristic curve for a sample from the third set of samples, and in figure 7 is given the characteristic curve for a sample from fourth set of samples.

In tables 1 and 2 are presented experimental values obtained for the four sets of samples.

The elasticity modulus calculated with (24) is:
- for the material from polyester resin reinforced with fiber glass (set of sample 2)

\[ E_L = 15124 \text{ MPa} \]

- for the material from polyester resin reinforced with carbon fiber (set of sample 3)

\[ E_L = 29536 \text{ MPa} \]

- for the set of sample 4

\[ E_L = 22330 \text{ MPa} \]

At the set of sample 4 we considered \( E_1 \) and \( E_2 \) as the theoretical elasticity modules, calculated for the sets of samples 2 and 3.

Conclusions

In the case of sandwich bars, with layers disposed symmetrically to the median plane, a request of traction leads to a symmetrical distribution of the stress.

In the case of bars built from three-layers, we can see that the variation of the elasticity modulus is almost linear with the volumetric proportion of median layer. The variation is even linear in the case in which transversal contraction coefficients for the median layer and outer layers are equal. If transversal contraction coefficients for the three layers are different then the variation of elasticity modulus of the composite, according to the volumetric proportion of median layer, is nonlinear. Graphics representations shows however, that we can apply the simplified relationship, deviations from linearity being very small.

Constitutive equation for composite materials in the form of sandwich bars with symmetrical distribution of the layers can be obtained by eliminating the parameter \( x \). In general case the constitutive equation is nonlinear. If in the relations which contain the tension tensor components and deformation tensor components, are neglected the higher order terms, then we obtain a constitutive equation which is linear.

References


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